

# Quantum Gravity Corrected Geodesic Motion and Violations of Equivalence Principle

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**Abstract:** Quantum Gravity theories predict a Generalized Uncertainty Principle (GUP) with a minimum length scale. We derive a deviation in the geodesic motion in a non-commutative geometry setting, the latter being compatible with GUP. The Equivalence Principle is violated since the GUP effect enters through the parameter  $(\beta m^2)/2$  with  $\beta$  being the GUP parameter ( $\sim$  Planck scale) and  $m$  the particle mass. The predicted violations are compared to experimental observations for Gravitational Redshift, Law of Reciprocal Action and Universality of Free Fall. The bounds derived for  $\beta$  are tighter than those obtained from quantum mechanical predictions given in [2] Phys.Rev.Lett.101:221301 (2008).

Quantum Gravity theories predict a Generalized Uncertainty Principle (GUP) with a minimum length scale. Is GUP compatible with Equivalence Principle (EP)? In the present paper we argue that there is a clash between GUP and EP. This is interesting since GUP is favored by String Theory [1], (as it introduces a short distance scale, conventionally taken as the Planck length), but at the same time String Theory in the low energy limit yields Einstein's General Relativity (that is based on EP). Indeed, the present day experimental accuracy is far from adequate to observe the mismatch conclusively but even a theoretical possibility of the above scenario is disturbing. On the other hand, as pointed out in [2], theoretical predictions of GUP effects can provide large upper bounds for  $\beta_0$ , (the GUP parameter  $\beta = \beta_0/M_{Planck}^2$ ), consistent with present day experimental observations and suggest the existence of a new scale between electroweak scale and Planck scale. In the present paper we derive bounds for  $\beta_0$  completely within the classical scenario of geodesic

motion in General Relativity. *Quite interestingly these General Relativity based bounds are closely comparable with the bounds in [2] derived in purely quantum mechanical setting.* In fact we have provided improved upper bounds for  $\beta_0$ .

We use the approach of [3] and work in Non-Canonical (NC) phase space framework that induces the GUP and derive, from first principles, the geodesic equation satisfied by a point particle living in the NC space. Working in weak linearized gravity and Newtonian low energy limit we show that in the leading order of NC parameter  $\beta$  the modified geodesic equation depends on the particle mass. Clearly this is a violation of the EP. So far the GUP oriented studies have been mostly kinematical but to analyze the dynamics it is essential to have a Lagrangian/Hamiltonian framework. In a recent paper [4] we have provided some examples of non-canonical relativistic particle Lagrangians that are compatible with GUP algebra. It is straightforward to generalize these models to interacting ones with gravitational or gauge interactions [4, 5]. There is some ambiguity involved in the explicit form of the generalized particle model because there can be many inequivalent extensions all of which induce GUP-type phase space and also reduce to the canonical particle model for  $\beta = 0$ . Only experimental results can distinguish one model from the other. In this perspective we have chosen the simplest model from the point of view of NC algebra that yields GUP.

Previous attempts to show EP violation from GUP [6] were restricted to Newtonian physics without considering the geodesic deviation. The latter was treated in [7] in a  $\kappa$ -Minkowski spacetime that is different from ours. Our NC framework is simpler since the coordinates commute even in presence of gravity but it retains the GUP induced minimal length feature.

The free GUP particle model in flat spacetime [4]

$$L = -A\eta_{\mu\nu}x^\mu\dot{p}^\nu + \beta(xp)(p\dot{p}) \quad (1)$$

with  $A = 1 - \beta\frac{p^2}{2}$ ,  $(ab) = \eta_{\mu\nu}a^\mu b^\nu$  satisfies the NC algebra,

$$\{x^\mu, p^\nu\} = -\left[\frac{g_{\mu\nu}}{\left(1 - \frac{\beta p^2}{2}\right)} + \frac{\beta p_\mu p_\nu}{\left(1 - \frac{3\beta p^2}{2}\right)\left(1 - \frac{\beta p^2}{2}\right)}\right], \quad \{x^\mu, x^\nu\} = \{p^\mu, p^\nu\} = 0. \quad (2)$$

In the presence of gravity, this is generalized to  $(\eta_{\mu\nu} \rightarrow g_{\mu\nu})$ ,

$$L = -Ag_{\mu\nu}x^\mu\dot{p}^\nu - \partial_\lambda g_{\mu\nu}p^\mu x^\nu \dot{x}^\lambda + \beta(xp)(p\dot{p}). \quad (3)$$

This is a first order system with constraints and the Dirac Hamiltonian scheme [8] is used to obtain the Dirac Brackets to first order in  $\beta$ ,

$$\begin{aligned} \{x^\mu, x^\nu\} &= 0 ; \quad \{p^\mu, p^\nu\} = Q^{\mu\nu} + \beta(H^{\mu\nu} - Q^{\mu\lambda}M_{\lambda\sigma}g^{\sigma\nu} - g^{\mu\lambda}M_{\sigma\lambda}Q^{\sigma\nu}), \\ \{x^\mu, p^\nu\} &= A^{-1}g^{\mu\nu} + \beta(cp^\mu p^\nu - g^{\mu\lambda}M_{\lambda\sigma}g^{\sigma\nu}) \end{aligned} \quad (4)$$

where the abbreviations are,

$$\begin{aligned} c &= \frac{\beta}{A(A - \beta p^2)}, \quad Q^{\alpha\lambda} = g^{\alpha\mu}g^{\lambda\nu}(\partial_\mu g_{\nu\sigma} - \partial_\nu g_{\mu\sigma})p^\sigma, \\ \beta H^{\alpha\lambda} &= \beta\left(\left(\frac{p^2}{2}g^{\alpha\mu} + p^\alpha p^\mu\right)g^{\nu\lambda} + \left(\frac{p^2}{2}g^{\nu\lambda} + p^\nu p^\lambda\right)g^{\alpha\mu}\right)(\partial_\mu g_{\nu\sigma} - \partial_\nu g_{\mu\sigma})p^\sigma, \\ M_{\mu\nu} &= -\left(\frac{1}{2}g_{\alpha\beta}\partial_\nu g_{\mu\lambda} + \frac{1}{2}g_{\mu\lambda}\partial_\nu g_{\alpha\beta} + g_{\alpha\mu}\partial_\nu g_{\lambda\beta} + g_{\alpha\lambda}\partial_\nu g_{\mu\beta}\right)p^\alpha p^\beta x^\lambda. \end{aligned} \quad (5)$$

This non-canonical algebra appears as Dirac Brackets. (Some details of the computation are provided in the Appendix.) For  $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$  the flat space GUP model (1,2) is recovered. The curvature corrected GUP algebra is a new extension, similar to the  $U(1)$  interaction extension discussed in [5].

The Hamiltonian equations of motion are obtained from,

$$\dot{x}^\mu = \{x^\mu, H\} = g_{\nu\lambda}p^\lambda\{x^\mu, p^\nu\} ; \quad \dot{p}^\mu = \frac{1}{2}p^\nu p^\lambda\{p^\mu, g_{\nu\lambda}\} + g_{\nu\lambda}p^\lambda\{p^\mu, p^\nu\}, \quad (6)$$

with the Hamiltonian constraint given by,

$$H = \frac{1}{2}(g_{\mu\nu}p^\mu p^\nu - m^2). \quad (7)$$

So far we have not done any approximation regarding  $g_{\mu\nu}$  but let us now linearize the gravitational interaction with

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2) ; \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2). \quad (8)$$

This yields slightly simplified forms of the equations of motion:

$$\dot{p}^\mu = \eta^{\mu\nu} \left( \frac{1}{2} \partial_\nu h_{\rho\sigma} - \partial_\sigma h_{\nu\rho} \right) p^\rho p^\sigma + \beta (2m^2 \eta^{\mu\nu} + p^\mu p^\nu) \left( \frac{1}{2} \partial_\nu h_{\rho\sigma} - \partial_\sigma h_{\nu\rho} \right) p^\rho p^\sigma, \quad (9)$$

$$\dot{x}^\mu = p^\mu + \beta (\eta_{\nu\rho} + h_{\nu\rho}) H^{\mu\nu} p^\rho. \quad (10)$$

Keeping the level of approximation in mind we can invert the above equations to get a modified geodesic equation,

$$\ddot{x}^\mu = \left[ \left( 1 + \frac{5}{2} \beta m^2 \right) \eta^{\mu\nu} \dot{x}^\rho \dot{x}^\sigma - \beta \frac{m^2}{2} \dot{x}^\mu (\dot{x}^\sigma \eta^{\rho\nu} + \dot{x}^\rho \eta^{\sigma\nu}) \right] \left( \frac{1}{2} \partial_\nu h_{\rho\sigma} - \partial_\sigma h_{\nu\rho} \right). \quad (11)$$

For  $\beta = 0$  the geodesic equation is reproduced. In the above we have used the constraint  $p^2 = m^2$ . Presence of  $m$  signals EP violation.

We wish to predict terrestrially observable effects of EP violation in our model. This allows us make one further approximation: the low energy or Newtonian limit. Renaming the parameter  $\beta m^2/2 = \beta_m$  from (11) we find,

$$\frac{d^2 t}{d\tau^2} = \beta_m \left( \frac{dt}{d\tau} \right)^2 \frac{dx^i}{dt} \eta^{\mu\nu} \partial_i h_{\mu\nu}, \quad (12)$$

$$\begin{aligned} \frac{d^2 x^i}{dt^2} &= \frac{1}{2} (1 + 5\beta_m) \partial^i h_{00} - \beta_m \frac{dx^i}{dt} \frac{dx^j}{dt} (\eta^{00} \partial_j h_{00} + \eta^{kl} \partial_j h_{kl}) \\ &\approx \frac{1}{2} (1 + 5\beta_m) \partial^i h_{00}. \end{aligned} \quad (13)$$

In the last step the quadratic velocity term is dropped.

*Gravitational Redshift:* In the conventional case  $\beta_m = 0$  in (), from Newton's equation and gravitational potential at a distance  $r$  from a mass  $M$ ,

$$\frac{d^2 \mathbf{x}}{dt^2} = -\nabla \phi; \quad \phi = -\frac{GM}{r}, \quad (14)$$

one identifies  $h_{00} = -2\phi \rightarrow g_{00} = -(1 + 2\phi)$  (see eg. [9]). In the present case we have  $(1 + 5\beta_m)h_{00} = -2\phi$  so that  $h_{00} \approx -2(1 - 5\beta_m)\phi$  leading to  $g_{00} = -(1 + 2(1 - 5\beta_m)\phi)$ .

In order to experimentally measure Gravitational Redshift effect [9] one needs two observation points, say  $x_1, x_2$  and consider a given atomic transition. The ratio of frequencies  $\nu_2$  - light coming from  $x_2$  to  $x_1$ , and  $\nu_1$ , both observed at  $x_1$ , is

$$\frac{\nu(x_2)}{\nu(x_1)} = \left( \frac{g_{00}(x_2)}{g_{00}(x_1)} \right)^{\frac{1}{2}} = \left( \frac{1 + 2(1 - 5\beta_m)\phi(x_2)}{1 + 2(1 - 5\beta_m)\phi(x_1)} \right)^{\frac{1}{2}} \approx 1 + (1 - 5\beta_m)(\phi(x_2) - \phi(x_1)), \quad (15)$$

where the above expression is linearized in the last step. Hence for two clocks  $A$  and  $B$  [10], with  $(\beta_m)_A = (\beta m_A^2)/2, \dots$ , we will have

$$\frac{\nu_A(x_2)}{\nu_A(x_1)} \approx 1 + (1 - 5(\beta_m)_A)(\phi(x_2) - \phi(x_1)); \quad \frac{\nu_B(x_2)}{\nu_B(x_1)} \approx 1 + (1 - 5(\beta_m)_B)(\phi(x_2) - \phi(x_1)). \quad (16)$$

Combining the above expressions we obtain the all important result [10],

$$\left( \frac{\nu_A(x_2)}{\nu_B(x_2)} \right) \approx \{1 - 5((\beta_m)_A - (\beta_m)_B)\}(\phi(x_2) - \phi(x_1)) \left( \frac{\nu_A(x_1)}{\nu_B(x_1)} \right). \quad (17)$$

A mismatch of the frequency ratios will signal a violation of the EP. The best present day observational result is  $|\alpha_{Hg} - \alpha_{Cs}| \leq 5.10^{-6}$  [11] where  $\alpha_{Hg}$ ,  $\alpha_{Cs}$  stand for clock-dependent parameters for Mercury and Cesium (for details see [10, 11]). In our case  $\alpha_{Hg} \equiv 5\beta m_{Hg}^2$ ,  $\alpha_{Cs} \equiv 5\beta m_{Cs}^2$ . Conventionally one considers  $\beta = \beta_0/M_{Planck}^2$  [2] with  $\beta_0 \approx 1$ , in which case the mismatch will be  $\approx (m_{Hg}^2 - m_{Cs}^2)/M_{Planck}^2 \approx 10^{-32}$ . Indeed this signal is very small. Another interpretation [2] is to consider an upper bound for  $\beta_0$ :  $\beta_0 \leq (10^{-9}/10^{-25})^2 \cdot 10^{-6} \approx 10^{26}$ . This is below the upper bound of  $\beta_0 \leq 10^{34}$  compatible with the electroweak scale but much tighter than the bounds suggested in [2] from Lamb shift and Landau level measurements, but weaker than  $\beta_0 \leq 10^{21}$  derived from Scanning Tunneling Microscope current measurement [2].

*Law of Reciprocal Action:* The notion of distinct masses was introduced by Bondi where the (Newtonian) gravitational force law between two masses  $A, B$  is generalized to

$$m_{Ai}\ddot{\mathbf{x}}_A = Gm_{Ap}m_{Ba} \frac{\mathbf{x}_B - \mathbf{x}_A}{|\mathbf{x}_B - \mathbf{x}_A|^3}, \quad m_{Bi}\ddot{\mathbf{x}}_B = Gm_{Bp}m_{Aa} \frac{\mathbf{x}_A - \mathbf{x}_B}{|\mathbf{x}_B - \mathbf{x}_A|^3}. \quad (18)$$

In the above force law for  $A$  [10]  $m_{Ai}$  is the *inertial* mass,  $m_{Ap}$  is the *passive* mass and  $m_{Aa}$  is the *active* mass as they appear below:

$$m_{Ai}\ddot{\mathbf{x}} = m_{Ap}\nabla U(\mathbf{x}); \quad \nabla^2 U(\mathbf{x}) = 4\pi m_{Aa}\delta(\mathbf{x}).$$

The motion of the center of mass coordinate  $\mathbf{X} = (m_{Ai}\mathbf{x}_A + m_{Bi}\mathbf{x}_B)/(m_{Ai} + m_{Bi})$  is given by

$$\ddot{\mathbf{X}} = G \frac{m_{Ap}m_{Bp}}{m_{Ai} + m_{Bi}} C_{BA} \frac{\mathbf{x}_B - \mathbf{x}_A}{|\mathbf{x}_B - \mathbf{x}_A|^3}, \quad C_{BA} = \frac{m_{Ba}}{m_{Bp}} - \frac{m_{Aa}}{m_{Ap}}. \quad (19)$$

For  $C_{BA} \neq 0$  the center of mass will possess a self-acceleration [10]. In our formulation the potential and hence the active mass gets modified so that

$$\begin{aligned} C_{BA} &= \frac{m_{Ba}}{m_{Bp}} - \frac{m_{Aa}}{m_{Ap}} = \frac{(1 - 5(\beta_m)_B)m_{Bi}}{m_{Bi}} - \frac{(1 - 5(\beta_m)_A)m_{Ai}}{m_{Ai}} \\ &= 5((\beta_m)_B - (\beta_m)_A) = 5\beta_0 \frac{m_B^2 - m_A^2}{M_{Plank}^2}. \end{aligned} \quad (20)$$

Observation of no self-acceleration of the moon by Lunar Laser Ranging provides a bound  $|C_{Al-Fe}| \leq 7.10^{-13}$  [12, 10]. This provides a considerably tighter bound  $\beta_0 \leq 10^{19}$  than the one provided by Gravitational Redshift (see above) and other earlier bounds [2].

*Universality of Free Fall:* According to General Relativity the neutral free particles follow the geodesic and hence the motion is independent of the nature of the neutral particle. Its' validity is tested by experimentally measuring the Eotvos parameter  $\eta = \frac{g_A - g_B}{\frac{1}{2}(g_A + g_B)}$  where  $g_A, g_B$  are accelerations of two particles  $A$  and  $B$  in the “same” gravitational field. A non-zero  $\eta$  signals violation of Universality of Free Fall. But in the present case the active mass gets different corrections for  $A$  and  $B$  and in turn the gravitational field perceived by them is not the same. In the field of  $M$  the acceleration of  $A$  is  $g_A = (1 - 5(\beta_m)_A)g$  (and similarly for  $B$ ). Thus we find

$$\eta = \frac{(1 - 5(\beta_m)_A) - (1 - 5(\beta_m)_B)}{\frac{1}{2}(1 - 5(\beta_m)_A) + (1 - 5(\beta_m)_A)} \approx 5\beta_0(m_B^2 - m_A^2)/M_{Plank}^2. \quad (21)$$

Tosion pendulum results provide  $\eta \leq 2.10_{-13}$  [10] yielding once again  $\beta_0 \leq 10^{19}$ . It should be noted that the results will not hold for macroscopic bodies due to the restriction  $\beta_m \ll 1$ .

To conclude we have shown that the minimally extended point particle model satisfying GUP leads to a modified geodesic equation. In the low energy and weak gravity limit considered here, this effect translates in to a modified gravitational potential, the correction depending upon the test particle energy/mass. This leads to a violation in the Equivalence Principle. Results are predicted for the violation in the contexts of Gravitational Red Shift, Law of Reciprocal Action and Universality of Free Fall. Comparison with experimental results predict improved bounds for the GUP parameter.

*Appendix:* We briefly discuss steps leading to the Dirac Brackets. In the presence of a set of Second Class Constraints  $\psi_\mu$ , with non-singular constraint algebra matrix  $\{\psi_\mu, \psi_\nu\}$ , the Dirac bracket between two generic variables  $A$  and  $B$  is defined as

$$\{A, B\}_{DB} = \{A, B\} - \{A, \psi_\mu\} \{\psi_\mu, \psi_\nu\}^{-1} \{\psi_\nu, B\}. \quad (22)$$

We have dropped the subscript  $\{, \}_{DB}$  throughout.

From the Lagrangian the conjugate momenta and constraints  $\phi_\mu^1, \phi_\nu^1$  are obtained:

$$\pi_\mu^x = \frac{\partial L}{\partial \dot{x}^\mu} = -\partial_\mu g_{\alpha\beta} p^\alpha x^\beta ; \quad \pi_\mu^p = \frac{\partial L}{\partial \dot{p}^\mu} = -Ag_{\mu\nu} x^\nu + \beta(xp)p_\mu, \quad (23)$$

$$\phi_\mu^1 \equiv \pi_\mu^p + Ag_{\mu\nu} x^\nu - \beta(xp)p_\mu \approx 0 ; \quad \phi_\mu^2 \equiv \pi_\mu^x + \partial_\mu g_{\alpha\beta} p^\alpha x^\beta \approx 0. \quad (24)$$

The following algebra shows that the constraints are Second Class,

$$\{\phi_\mu^1, \phi_\nu^1\} = 0, \quad \{\phi_\mu^2, \phi_\nu^2\} = (\partial_\mu g_{\nu\alpha} - \partial_\nu g_{\mu\alpha}) p^\alpha, \quad \{\phi_\mu^1, \phi_\nu^2\} = Ag_{\mu\nu} - \beta p_\mu p_\nu + \beta M_{\mu\nu}, \quad (25)$$

The constraint matrix,

$$\begin{aligned} \{\phi_\mu^i, \phi_\nu^j\} &= \begin{bmatrix} 0 & (Ag_{\mu\nu} - \beta p_\mu p_\nu) + \beta M_{\mu\nu} \\ -(Ag_{\mu\nu} - \beta p_\mu p_\nu) + \beta M_{\nu\mu} & (\partial_\mu g_{\nu\alpha} - \partial_\nu g_{\mu\alpha}) p^\alpha \end{bmatrix} \\ &\equiv A + \beta B \end{aligned} \quad (26)$$

with  $A$  and  $B$ ,

$$A = \begin{bmatrix} 0 & (Ag_{\mu\nu} - \beta p_\mu p_\nu) \\ -(Ag_{\mu\nu} - \beta p_\mu p_\nu) & (\partial_\mu g_{\nu\alpha} - \partial_\nu g_{\mu\alpha}) p^\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \beta M_{\mu\nu} \\ -\beta M_{\nu\mu} & 0 \end{bmatrix}, \quad (27)$$

yields the inverse,

$$(A + \beta B)^{-1} \approx A^{-1} - \beta A^{-1} B A^{-1}, \quad (28)$$

to first order in  $\beta$ . The Dirac Brackets are computed from (22).

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